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Dynamical Relaxation and Universal Short-Time Behavior of Finite Systems

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A system belonging to the dynamic universality class of model A is considered in a block ($V = L^d$) geometry with periodic boundary conditions. The relaxation of the order parameter $m(t)$ from an initial value $m^{(i)}$ is investigated at the bulk critical temperature. We demonstrate that a proper scaling description of the problem involves two characteristic times, $t_L \sim L^z$ and $t_i \sim [m^{(i)}]^{-z/x_i}$, where z is the familiar dynamic bulk exponent, while x_i is an independent new bulk exponent discovered recently. Previous analyses of the problem either were restricted to $t \gg t_i$, or tacitly used the incorrect assumption that $x_i = \beta/\nu$. Thus the short-time regime $t \ll t_i$ with universal dependence on $m^{(i)}$ was missed. As a concrete example we study the exact solution in the large-n limit.

KEY WORDS: Dynamic critical phenomena; finite size; scaling; large-n limit.

1. INTRODUCTION

Consider a macroscopic ferromagnet at a temperature T slightly above its Curie temperature T_c and in zero magnetic field H. Assume that this ferromagnet is brought into a nonequilibrium state such that the magnetization density $m(\mathbf{x}, t)$ at initial time $t_0 = 0$ has the initial value $m^{(i)}(\mathbf{x}) \equiv m(\mathbf{x}, t_0)$. For simplicity, we will take the initial (coarse-grained) density to be uniform, so that $m^{(i)} \equiv m^{(i)}(\mathbf{x})$ coincides with the initial value $m(t_0)$ of the bulk order parameter $m(t)$ (total magnetization per unit volume). Such an initial configuration can easily be prepared in computer simulations. In real experiments this could be achieved by first preparing the system in a thermal equilibrium state at a temperature $T_i > T_c$ and a magnetic field $H_i > 0$ and then rapidly changing these thermodynamic fields to values $T = T_c$ and $H = 0$. Supposing the system is ergodic and satisfies

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the usual requirements of detailed balance and relaxation toward thermal equilibrium, it will evolve from this initial state toward the unique Gibbs state pertaining to the given values of T and H. Accordingly, $m(t)$ must relax to zero. To simplify our subsequent considerations, we shall restrict our attention to ferromagnets whose critical dynamics belongs to the universality class of the n-component model called A in the Halperin–Hohenberg–Ma classification.[1] A well-known example with $n = 1$ is the fully anisotropic Heisenberg ferromagnet.

For a long time the accepted picture of the ensuing relaxation process was the following.[2–6] For times up to a microscopic time scale t_{mic}, the behavior of $m(t)$ (as of other quantities) depends on microscopic details and is thus *nonuniversal.* On the other hand, in the extreme long-time regime $t \gtrsim t_\tau$, the order parameter decays as $m(t) \sim \exp(-t/t_\tau)$ in a *universal* fashion (up to scales). This exponential decay defines the so-called *bulk relaxation time* t_τ, which varies as

$$t_\tau \sim \tau^{-\nu z} \tag{1}$$

near criticality, where $\tau \equiv (T - T_c)/T_c$ while z and ν are the usual dynamic bulk exponent and the correlation-length exponent, respectively. Finally, for times sufficiently long compared with t_{mic} but small compared to t_τ, there is a regime of *nonlinear relaxation* characterized by *algebraic time decay* of properties, such as the behavior $m(t) \sim t^{-\beta/\nu z}$ of the order parameter.

Recently, Janssen *et al.*[7] have called attention to the fact that there exists another regime in which the behavior is universal. They showed that the initial magnetization $m^{(i)} > 0$ provides a further time scale t_i, which varies as

$$t_i \sim [m^{(i)}]^{-z/x_i} \tag{2}$$

on long scales, where x_i, the scaling dimension of $m^{(i)}$, is a new critical exponent. For given not too large initial magnetization $m^{(i)}$ and sufficiently small reduced temperature τ, one has $t_{\text{mic}} \ll t_i \ll t_\tau$. Hence there is a time regime $t_{\text{mic}} \ll t \lesssim t_i$. As shown by Janssen *et al.*,[7] this features a *universal dependence* of quantities *on the initial conditions.* Specifically for the order parameter, this universal stage is marked by an *increase* of the form

$$m(t) \sim t^{\theta'} \tag{3}$$

with $\theta' > 0$. This phenomenon, which somewhat contradicts naive expectation, was termed *critical initial slip* by Janssen *et al.*[7] It can be understood by a careful analysis of the dependence on initial conditions and their behavior under renormalization-group transformations. The underlying reason is that $m^{(i)}$ has an independent scaling dimension x_i; this is generally different from $x_\phi \equiv \beta/\nu$, the scaling dimension of the order parameter in thermal equilibrium, with which $m(t)$ scales at large times. Simple scaling considerations (see below) or more sophisticated techniques[7] yield the relation

$$\theta' = (x_i - x_\phi)/z \tag{4}$$

Hence the initial rise (3) simply corresponds to the fact that $x_i > x_\phi$ for the type of systems considered. As can be seen from these considerations, initial conditions are in many respects similar to boundary conditions at surfaces for continuum field theories in bounded geometries.[8] The short-time singularity (3) is the analog of a short-distance singularity of the order-parameter profile near the surface of a semi-infinite system. By analogy with the mechanism producing the former singularity, the latter can be traced back to the fact that the magnetization densities at the surface and infinitely far away from it have distinct scaling dimensions. Let us also note that the problem of initial conditions arises just as well in a variety of other contexts, which have been studied by a number of authors.[9–13]

In the present paper we will investigate the relaxation process explained above in a system of finite spatial extent. Our aim is to incorporate a proper treatment of the initial condition into the theory of relaxation in finite-size systems. Previous analyses of relaxation in such systems were either restricted to those long-time regimes in which all dependence of physical quantities on the initial condition has already been lost, or assumed the scaling dimension x_ϕ for $m^{(i)}$, thus missing the initial-slip stage.[14]

The remainder of this paper is organized as follows. In the next section we introduce the model and discuss briefly how its field theory can be defined, recalling, in particular, how the initial condition may be implemented within the framework of a functional-integral formulation of the theory. In Section 3 a phenomenological scaling approach is applied to the relaxation process. The scaling expressions for the magnetization are discussed, the relevant time scales are identified, and the various asymptotic regimes are described. In Sections 4 and 5 the phenomenological results are confirmed by means of an explicit field-theoretic calculation. We choose the large-n limit of the n-vector model because in this case exact, largely analytic results can be derived.

2. THE MODEL

We consider a continuum field theory on the d-dimensional interval $V = [0, L]^d \subset \mathbb{R}^d$ whose dynamics is defined by the Langevin equation

$$\partial_t \phi(\mathbf{x}, t) = -\lambda \frac{\delta \mathscr{H}\{\phi\}}{\delta \phi(\mathbf{x},t)} + \zeta(\mathbf{x}, t) \tag{5}$$

with the Ginzburg–Landau Hamiltonian

$$\mathscr{H}\{\phi\} = \int_V \left[\frac{1}{2} (\nabla\phi)^2 + \frac{\tau}{2} \phi^2 + \frac{g}{4!\, n} (\phi^2)^2 \right] \tag{6}$$

where $\phi = (\phi_\alpha)$ is an n-component field, while $\zeta = (\zeta_\alpha)$ is a Gaussian random force with mean zero and variance

$$\langle \zeta_\alpha(\mathbf{x}, t)\, \zeta_\beta(\mathbf{x}', t') \rangle = 2\lambda \delta_{\alpha\beta}\, \delta(\mathbf{x} - \mathbf{x}')\, \delta(t - t') \tag{7}$$

We have scaled the coupling constant g by n in order to make the model well defined in the limit $n \to \infty$. Periodic boundary conditions will be assumed in all d coordinate directions, so that the topology becomes that of a d-torus.

The above equations define model A in a finite block geometry. This has been analyzed previously.[15–17,14] It satisfies the standard requirements of causality, detailed balance, and relaxation toward thermal equilibrium for models of critical dynamics. Accordingly, its stationary state is described by the Bolzmann factor $\exp(-\mathscr{H})$.

We wish to study the relaxation from an initial state, which we specify through a probability measure

$$dP_i\{\phi^{(i)}\} = \prod_{\mathbf{x}} P_{\mathbf{x}}[\phi^{(i)}(\mathbf{x})]\, d\phi^{(i)}(\mathbf{x}) \tag{8}$$

for the field $\phi^{(i)}(\mathbf{x}) \equiv \phi(\mathbf{x}, t_0)$. On physical grounds one would choose a distribution $P_{\mathbf{x}}$ centered around $m^{(i)}$ with some width w, e.g.,

$$P_{\mathbf{x}}(\varphi) = (2\pi w)^{-1/2}\, e^{-(\varphi - m^{(i)})^2/2w} \tag{9}$$

However, Janssen *et al.*[7] have shown that w corresponds to an irrelevant variable. Hence we may let w approach its fixed-point value 0, so that $P_{\mathbf{x}}$ becomes a δ-function.

In order to apply field-theoretic methods, it is convenient to use the path-integral representation[18] of the Langevin equation (5). With the imaginary-valued response field denoted by $\tilde{\phi}$, the generating functional of connected response and correlation functions takes the form

$$\mathscr{W}\{\tilde{J}, J\} = \ln \int \mathscr{D}(i\tilde{\phi}, \phi) \exp\left[-\mathscr{J}\{\tilde{\phi}, \phi\} - \mathscr{H}_i\{\phi^{(i)}\} + \int_0^\infty dt \int_V (J\phi + \tilde{J}\tilde{\phi})\right] \tag{10}$$

where the action functional is given by

$$\mathscr{J}\{\tilde{\phi}, \phi\} = \int_0^\infty dt \int_V \left[\tilde{\phi}\left(\partial_t \phi + \lambda(\tau - \Delta)\phi + \frac{1}{6n}\lambda g \phi^2 \phi \right) - \lambda \tilde{\phi}^2 \right] \tag{11}$$

and

$$\mathscr{H}_i\{\phi^{(i)}\} \equiv -\ln\left[\prod_{\mathbf{x}} P_{\mathbf{x}} + \text{const} \right] = \frac{1}{2w} \int_V [\phi(\mathbf{x}, 0) - m^{(i)}]^2 \tag{12}$$

is the contribution from the probability distribution of initial values. The functional measure $\mathscr{D}(i\tilde{\phi}, \phi)$, which in symbolic notation is proportional to

$$\prod_{\mathbf{x}, t} [i\, d\tilde{\phi}(\mathbf{x}, t)\, d\phi(\mathbf{x}, t)]$$

is understood to be defined using a prepoint discretization with respect to time.

The generating functional $\mathscr{W}_0\{\tilde{J}, J\}$ of the Gaussian theory with $g = 0$ is well defined and can be computed in a straightforward manner. The result is[7]

$$\mathscr{W}_0\{\tilde{J}, J\} = \int_0^\infty dt \int_0^\infty dt' \left[\tfrac{1}{2} J(t)\, C_0(t, t')\, J(t') + J(t)\, G_0(t, t')\, \tilde{J}(t')\right]$$
$$+ \int_0^\infty dt [J(t)\, G_0(t, 0) m^{(i)}] \tag{13}$$

where we have used a condensed notation in which C_0 and G_0 or J and $\tilde{J}$ are viewed as matrices or vectors, respectively, both with regard to component indices and position variables $\mathbf{x}$. The explicit expressions for the free propagators $C_0(t, t') \equiv \langle \phi(t)\, \phi(t') \rangle_0^C$ and $G(t, t') \equiv \langle \phi(t)\, \tilde{\phi}(t') \rangle_0^C$ (where C means cumulant or connected) will not be needed here. They may be gleaned from ref. 7, noting that the momentum-space results given in its Eqs. (3.5a) and (3.6a) for the infinite system carry over to the present finite-size case, except that the wavevectors $\mathbf{q}$ now are restricted to the discrete values $\mathbf{q} = (2\pi/L)\mathbf{m}$, $\mathbf{m} \in \mathbb{Z}^d$ (see also the analogous expressions presented in Section 4 for the full theory in the limit $n \to \infty$).

To define the generating functional of the full theory, we use

$$\mathscr{W}\{\tilde{J}, J\} = \ln[\exp(-\mathscr{I}_{\text{int}}\{\delta/\delta\tilde{J}, \delta/\delta J\}) \exp(\mathscr{W}_0\{\tilde{J}, J\})] \tag{14}$$

where $\mathscr{I}_{\text{int}}\{\tilde{\phi}, \phi\}$ is the interaction part of $\mathscr{I}$, and we regulate the ultraviolet singularities of the resulting perturbation series by means of a cutoff Λ, restricting the allowed momenta to values with $|\mathbf{q}| \leqslant \Lambda$.

Before we embark on details of the exact relaxation analysis in the limit $n \to \infty$, let us first discuss the problem on the level of the phenomenological scaling theory.

3. PHENOMENOLOGICAL SCALING ANALYSIS OF RELAXATION

Choosing the $\alpha = 1$ axis along the uniform initial magnetization density $m^{(i)}$, we have

$$\langle \phi_\alpha(\mathbf{x}, t) \rangle = \delta_{\alpha 1}\, n^{1/2} m(t) \tag{15}$$

where the right-hand side is independent of $\mathbf{x}$ by translational invariance. Let us assume that d is less than $d^* = 4$, the upper critical dimension, so that hyperscaling is valid and $x_\phi = \beta/\nu$. In accordance with familiar renormalization-group ideas for systems of finite size[19] and ref. 7, it is then natural to assume that sufficiently close to criticality we have

$$m(t, \tau, L, m^{(i)}, g) \approx C_m l^{-\beta/\nu} m^*(l^{-z} C_t t, l^{1/\nu} C_\tau \tau, l^{-1} L, l^{x_i} C_i m^{(i)}) \tag{16}$$

where $l \gg 1$ is the spatial rescaling factor and m^* means m with all irrelevant scaling fields set to zero. (In particular, g is set to its fixed-point value g^*.) The metric factors C_m, C_t, C_τ, and C_i are the only nonuniversal, system-dependent parameters; the function m^* is universal. In this section all dimensionfull quantities are assumed to be scaled by appropriate powers of a momentum scale μ and a frequency scale λ, respectively.

Upon choosing $l = L$ one arrives at the scaling expression

$$m(t, \tau, L, m^{(i)}) \approx C_m L^{-\beta/\nu} \mathscr{Y}(L^{-z} C_t t, L^{1/\nu} C_\tau \tau, L^{x_i} C_i m^{(i)}) \tag{17}$$

where we have suppressed the variable g on the left-hand side. The function $\mathscr{Y}$ has a finite and nonzero limit

$$\mathscr{Y}_c(y_t, y_i) \equiv \lim_{y_\tau \to 0^+} \mathscr{Y}(y_t, y_\tau, y_i) \tag{18}$$

At bulk criticality ($T = T_c$, with $H = 0$), (17) reduces to

$$m_c(t, L, m^{(i)}) \approx C_m L^{-\beta/\nu} \mathscr{Y}_c(L^{-z} C_t t, L^{x_i} C_i m^{(i)}) \tag{19}$$

In order to exhibit the various time scales, one can rewrite (17) in the equivalent form [corresponding to the choice $\ell = (C_t t)^{1/z}$ in (16)]

$$m(t, \tau, L, m^{(i)}) \approx C_m (C_t t)^{-\beta/\nu z} \mathscr{X}(t/t_\tau, t/t_L, t/t_i) \tag{20}$$

with

$$t_\tau = C_t^{-1} (C_\tau \tau)^{-\beta/\nu z} \tag{21a}$$

$$t_L = C_t^{-1} L^z \tag{21b}$$

and

$$t_i = C_t^{-1} [C_i m^{(i)}]^{-z/x_i} \tag{21c}$$

By analogy with above, the limit $\tau \to 0^+$ can be taken in (20) since

$$\mathscr{X}_c(\vartheta_L, \vartheta_i) \equiv \mathscr{X}(0^+, \vartheta_L, \vartheta_i) \tag{22}$$

exists and is nonvanishing.

We can also take the bulk limit $L \to \infty$ in (20). This yields the scaling expression

$$m_b(t, \tau, m^{(i)}) \approx C_m (C_t t)^{-\beta/\nu z} \mathscr{X}_b(t/t_\tau, t/t_i) \tag{23}$$

for the bulk magnetization m_b, where

$$\mathscr{X}_b(\vartheta_\tau, \vartheta_i) \equiv \lim_{\vartheta_L \to 0^+} \mathscr{X}(\vartheta_\tau, \vartheta_L, \vartheta_i) \tag{24}$$

At bulk criticality this becomes

$$m_{bc}(t, m^{(i)}) \approx C_m (C_t t)^{-\beta/\nu z} \mathscr{X}_{bc}(t/t_i) \tag{25}$$

with

$$\mathscr{X}_{bc}(\vartheta_i) \equiv \lim_{\vartheta_\tau \to 0^+} \mathscr{X}_b(\vartheta_\tau, \vartheta_i) \tag{26}$$

In general, attention must be paid to the order of limits. However, in this case we expect the limiting function $\mathscr{X}_{bc}(\vartheta_i)$ to agree with $\mathscr{X}_{cb}(\vartheta_i) \equiv \lim_{\vartheta_L \to 0^+} \mathscr{X}_c(\vartheta_L, \vartheta_i)$.

In order that $m_{bc} \sim m^{(i)}$ as $t \to 0$, the scaling function $\mathscr{X}_{bc}$ must have a short-time singularity of the form

$$\mathscr{X}_{bc}(\vartheta_i) \underset{\vartheta_i \to 0^+}{\approx} \mathscr{X}_{bci} (\vartheta_i)^{x_i/z} \tag{27}$$

where $\mathscr{X}_{bci}$ is a universal constant. As discussed by Janssen *et al.*,[7] this short-time singularity can be corroborated by means of an operator-product expansion analogous to the one used to determine the behavior of the order-parameter profile in semi-infinite systems at short distances from the surface.[20,8]

For the explicit computational analysis presented in Sections 4 and 5 still other, yet equivalent, scaling forms of m_c and m_{bc} will be most convenient for studying the influence of the initial condition at bulk criticality. The forms we shall prefer there are

$$m_c(t, L, m^{(i)}) \approx C_m (C_i m^{(i)})^{\beta/\nu x_i} \mathscr{Z}_c(t/t_i, t_i/t_L) \tag{28}$$

and

$$m_{bc}(t m^{(i)}) \approx C_m (C_i m^{(i)})^{\beta/\nu x_i} \mathscr{Z}_{bc}(t/t_i) \tag{29}$$

where $\mathscr{Z}_{bc}(\vartheta_i) \equiv \mathscr{Z}_c(\vartheta_i, 0^+)$, while t_L and t_i are given in (21b) and (21c), respectively.

Previous scaling analyses[14,15,3-5] have focused on regimes with $\vartheta_i \equiv t/t_i \gg 1$. To make contact with these, we need the corresponding limiting behavior of the scaling functions $\mathscr{Y}$ and $\mathscr{X}$. Upon taking the limit $t_i \to 0$ (i.e., $m^{(i)} \to \infty$) with the other variables fixed, we see that this is described by the scaling functions

$$\mathscr{Y}_\infty(y_t, y_\tau) \equiv \lim_{y_i \to \infty} \mathscr{Y}(y_t, y_\tau, y_i) \tag{30}$$

and

$$\mathscr{X}_\infty(\vartheta_\tau, \vartheta_L) \equiv \lim_{\vartheta_i \to \infty} \mathscr{X}(\vartheta_\tau, \vartheta_L, \vartheta_i) \tag{31}$$

respectively, where in accordance with all previous work cited above, we take it for granted that these limits exist. In $\mathscr{X}_\infty$, the bulk limit can again easily be taken to obtain the scaling function

$$\mathscr{X}_{b\infty}(\vartheta_\tau) \equiv \mathscr{X}(\vartheta_\tau, 0^+, \infty) \tag{32}$$

whose limiting behavior at small and large values of ϑ_τ is given by

$$\lim_{\vartheta_\tau \to 0^+} \mathscr{X}_{b\infty}(\vartheta_\tau) \equiv \mathscr{X}_{cb\infty} \tag{33}$$

and

$$\mathscr{X}_{b\infty} \underset{\vartheta_\tau \to \infty}{\sim} e^{-\vartheta_\tau} \tag{34}$$

respectively.

We are now ready to discuss the various asymptotic regimes. At bulk criticality, the system has two characteristic (macroscopic) time scales, t_i and t_L as defined above. As a consequence, we have to distinguish between two main asymptotic types of relaxational behavior according to whether $t_i \ll t_L$ (case I) or $t_i \gg t_L$ (case II).

Case I may be called asymptotic bulk case, as finite-size effects become important only for late times. The initial-slip stage $m_c(t) \sim t^{\theta'}$ for $t \ll t_i$ is followed by a crossover at time $t \approx t_i$ to the bulk behavior in the regime $t_i \ll t \ll t_L$, in which the magnetization decays as $m_c(t) \sim t^{-\beta/\nu z}$. In the latter regime any trace of the initial conditions has been lost. Finally, at $t \approx t_L$, the system crosses over to the usual finite-size relaxation behavior $m_c(t) \sim \exp(-t/t_L)$.

Case II will be called the asymptotic finite-size case, as finite-size effects are important during most of the relaxational process. Only for very early times, when t is much less than t_L, do we expect to see initial-slip behavior, governed by the bulk exponent θ'. For $t \approx t_L$, the system enters a new universal regime, where we expect to find essentially exponential decay of the magnetization with yet unknown dependence on the initial conditions.

4. SHORT-TIME BEHAVIOR IN BULK SYSTEM

In the following we concentrate on the large-n limit of the n-vector model at the bulk critical temperature. First we derive in this section results for the infinite-volume limit, some of which were partly derived in ref. 7. Afterward these will be extended to the finite-size case. In the limit $n \to \infty$ all higher-order cumulants factorize into products of two-point functions. Thus the model becomes Gaussian with a time-dependent shifted temperature $\tau(t)$.

The self-consistent set of equations that determines $\tau(t)$ and $m(t)$ and which will be solved exactly below reads

$$\partial_t m(t) = -\lambda \tau(t)\, m(t) \tag{35}$$

and

$$\tau(t) = \frac{g}{6}\,[C(t) - C(\infty) + m^2(t)] \tag{36}$$

where

$$C(t) = \int_{\mathbf{q}}^{\Lambda} C(q; t, t) \qquad \text{with} \quad \int_{\mathbf{q}}^{\Lambda} \equiv \frac{1}{(2\pi)^d} \int_{|\mathbf{q}| < \Lambda} d^d q \tag{37}$$

Since we are mainly interested in the time dependence for the moment, all arguments of $m(t)$ except t are omitted. The (Fourier-space) correlation function $C(q; t, t)$ is related to the response propagator by

$$C(q; t, t') = 2\lambda \int_0^\infty dt''\, G(q; t, t'')\, G(q; t', t'') \tag{38}$$

and the response propagator is given by

$$G(q; t, t') = \theta(t-t') \exp\left\{-\lambda(t-t')\, q^2 - \lambda \int_{t'}^{t} dt''\, \tau(t'')\right\} \tag{39}$$

Known general properties of the model (detailed balance, etc.) ensure that the system must relax in the limit $t \to \infty$ toward the thermal equilibrium state. In the present case this is the critical state. Hence the requirement that $m^2(\infty) = \tau(\infty) = 0$ implies the choice $C(\infty) = \int_{\mathbf{q}}^{\Lambda} 1/q^2$ in (36).

When Eqs. (35) and (36) are combined, τ can be eliminated, and one obtains a linear integrodifferential equation for $f(t) \equiv 1/m^2(t)$ that can be solved by Laplace transformation.[21] The result for the Laplace transform $\tilde{f}(s) \equiv \int_0^\infty dt\, e^{-st} f(t)$ can be written as

$$\tilde{f}(s) = \frac{s/[m^{(i)}]^2 + \lambda g/3}{1 + (g/6)\, I_\Lambda(s/2\lambda)}\, s^{-2} \tag{40}$$

with

$$I_\Lambda(a) = \int_{\mathbf{q}}^{\Lambda} \frac{1}{q^2(q^2+a)} \tag{41}$$

The integral $I_\Lambda(a)$ with $a > 0$ is finite in dimensions $d > 2$, whenever $\Lambda < \infty$. Furthermore, it has a finite limit $\Lambda \to \infty$, if $d < 4$. For $2 < d < 4$, one has

$$I_\Lambda(a) = A_\varepsilon a^{-\varepsilon/2} - \Lambda^{-\varepsilon} K_d\, g_\varepsilon(a/\Lambda^2). \tag{42}$$

Here $\varepsilon \equiv 4 - d$,

$$A_\varepsilon = \frac{1}{2} K_d \Gamma\left(\frac{d-2}{2}\right) \Gamma\left(\frac{4-d}{2}\right) = -(4\pi)^{-d/2}\, \Gamma\left(\frac{2-d}{2}\right) \tag{43}$$

and K_d means as usual the surface area of a d-dimensional unit sphere divided by $(2\pi)^d$. The cutoff-dependent part depends on the specific cutoff procedure employed; for our (sharp-cutoff) procedure we find

$$g_\varepsilon(a/\Lambda^2) = (1/\varepsilon)\, {}_2F_1(1, \varepsilon/2; 1+\varepsilon/2; -a/\Lambda^2) \tag{44}$$

where ${}_2F_1$ is the hypergeometric function. In the infrared regime of interest $\hat{s} \equiv s/2\lambda\Lambda^2 \ll 1$, we may use the expansion

$$g_\varepsilon(\hat{s}) = \frac{1}{\varepsilon}\,[1 + O(\hat{s})] \tag{45}$$

Substitution of these results into (40) yields for small $\hat{s}$ the behavior

$$\tilde{f}(s = \hat{s}2\lambda\Lambda^2) = \tilde{f}_{\text{as}}(s)[1 + C_u(u - u^*)\,\hat{s}^{\varepsilon/2} + O(\hat{s}^\varepsilon)] \tag{46}$$

with

$$\tilde{f}_{\text{as}}(s) = (2\lambda)^{-1}\Lambda^{-4+\varepsilon}A_\varepsilon^{-1}\hat{s}^{-2+\varepsilon/2}(1 + D_u\hat{s}\Lambda^{2-\varepsilon}/[m^{(i)}]^2), \qquad D_u = 6/u \tag{47}$$

and $u^* = 6\varepsilon/K_d$, where u is the dimensionless coupling constant

$$u \equiv g\Lambda^{-\varepsilon} \tag{48}$$

while D_u and $C_u = K_d/(uA_\varepsilon\varepsilon)$ are nonuniversal constants.

These results exhibit several expected features. First, the asymptotic part $\tilde{f}_{\text{as}}$ takes a scaling form, which is universal up to nonuniversal (i.e., g and Λ dependent) metric factors. The fact that $m^{(i)}$ enters in the combination $\hat{s}/[m^{(i)}]^2$ tells us that the exponent z/x_i introduced in (2), which describes how t^{-1} and hence s scales with $m^{(i)}$, is to be identified as $z/x_i = 2$. Use of the familiar value $z = 2$ of the dynamic exponent z then yields

$$x_i = 1 \tag{49}$$

Second, the leading corrections to scaling are down by a factor $s^{\omega/z}$ with $\omega = \varepsilon$, the value of the correction-to-scaling exponent ω known from the static theory. Third, the corrections to scaling are seen to vanish for the special value $u = u^*$.

In order to make contact with the work of Janssen *et al.*,[7] it will be helpful to indicate how these findings can be phrased in the language of renormalized field theory. As usual, the critical behavior of $\tilde{f}(s)$ can be extracted by studying the limit $\Lambda \to \infty$. Yet this limit cannot be taken naively, because Λ not only serves as a cutoff, but also is the only inverse length remaining at criticality. To disentangle these two distinct roles of Λ in renormalized field theory, one introduces an arbitrary reference momentum μ and reparametrizes the (cutoff-) regularized bare theory such that a meaningful renormalized theory results in the limit $\Lambda \to \infty$. In the present case, the order-parameter field $\phi(\mathbf{x}, t > 0)$ scales with its naive dimension

$(d-2)/2$ and hence needs no reparametrization. However, the scaling dimension of $m^{(i)}$ differs from its naive one: Writing

$$x_i = \frac{d-2+\eta_i}{2} \tag{50}$$

we see from (49) that the anomalous dimension $\eta_i/2$ is given by[2]

$$\eta_i = \varepsilon \tag{51}$$

Accordingly the renormalized initial magnetization $[m^{(i)}]_{\text{ren}}$ should be introduced through a reparametrization of the form

$$m^{(i)} = Z_i^{1/2} [m^{(i)}]_{\text{ren}} \tag{52}$$

with a renormalization factor $Z_i = Z_i(u, \Lambda/\mu)$ that varies as

$$Z_i \sim (\mu/\Lambda)^{\eta_i} \tag{53}$$

in the large-cutoff limit. With such a reparametrization the renormalized theory yields the asymptotic expression for $\tilde{f}(s)$ given in (47), except that $[m^{(i)}]_{\text{ren}}$ appears in place of $m^{(i)}$ and the reference unit is μ rather than Λ. These findings are in conformity with those of Janssen *et al.*[7]

In the following we will ignore corrections to scaling, using the asymptotic scaling form (47) of $\tilde{f}(s)$. For notational simplicity, we set the momentum unit Λ in (47) equal to one. The Laplace backtransform of this equation can be computed analytically with the result

$$m_{bc}(t) = \left[A_\varepsilon \frac{\Gamma(1-\varepsilon/2)}{D_u} \right]^{1/2} m^{(i)} \, (2\lambda t)^{\varepsilon/4} \left\{ 1 + \frac{4\lambda}{D_u(2-\varepsilon)} \, t [m^{(i)}]^2 \right\}^{-1/2} \tag{54}$$

Recalling the well-known $n = \infty$ values $\beta = 1/2$ and $1/\nu = 2 - \varepsilon$ of the static critical exponents, one sees that the magnetization $m_{bc}(t)$ has indeed the scaling form (29). A convenient choice of the nonuniversal time scale t_i is

$$t_i = \frac{D_u(2-\varepsilon)}{4\lambda} \frac{1}{[m^{(i)}]^2} \tag{55}$$

Upon appropriate normalization of its amplitude, the scaling function $\mathscr{X}_{bc}$ becomes

$$\mathscr{X}_{bc}(\vartheta_i) = \frac{\vartheta_i^{\varepsilon/4}}{(1+\vartheta_i)^{1/2}} \tag{56}$$

[2]The exponent η_i introduced here differs from the exponent η_0 used by Janssen *et al.*[7] The latter exponent is twice the anomalous dimension of $\tilde{\phi}|_{t=0}$, the initial response field, and related to η_i via $\eta_0 = 4 - 2z - \eta - \eta_i$, i.e., $\eta_0 = -\eta_i$ in the $n = \infty$ case studied here.

According to the phenomenological scaling theory of Section 3, $\mathscr{X}_{bc}$ should vary as $\vartheta_i^{\theta'} \to 0$. The $n=\infty$ value $\varepsilon/4$ of θ' obtained here is in conformity with (4) and the $n=\infty$ value (49) of x_i. Likewise, the exponent that governs the asymptotic behavior of $\mathscr{X}_{bc} \sim \vartheta_i^{-\beta/vz}$ for large ϑ_i is seen to take its familiar $n \to \infty$ value $\beta/vz = (2-\varepsilon)/4$.

These findings confirm that the order parameter runs through two universal regimes during the relaxation process. In the early stage, $t \ll t_i$, the initial-slip behavior $m_{bc}(t) \sim (t/t_i)^{\theta'}$ is observed. Then the process crosses over to the familiar long-time behavior $m_{bc}(t) \sim t^{-\beta/vz}$, thus losing its dependence on the initial magnetization.

5. FINITE-SIZE RESULTS

Having discussed the bulk case, we next wish to extend the above analysis to the finite-size case $L < \infty$. As before, we restrict ourselves to studying the relaxation process at the bulk critical point. Owing to the simple hypercubical geometry with periodic boundary conditions chosen in (6), the set of self-consistent equations one obtains in the limit $n \to \infty$ remains to be given by (35)–(39), except that the momentum integration $\int_{\mathbf{q}}^{\Lambda}$ is to be replaced by a sum $\sum_{\mathbf{q}}$ over discrete momenta $\mathbf{q} = (2\pi/L)\mathbf{m}$ with $\mathbf{m} \in \mathbb{Z}^d$ (and $|\mathbf{q}| < \Lambda$). The mass counterterm used in (36), namely the bare equilibrium mass at the bulk critical point, remains the same. This implies that, at bulk criticality, $\tau(t)$ is positive for finite L and $t \to \infty$, and approaches zero for $L \to \infty$.

The sum over discrete modes may be conveniently rewritten with the aid of Poisson's identity

$$\frac{1}{L^d} \sum_{\mathbf{m} \in \mathbb{Z}^d} \delta\left(\mathbf{q} - \frac{2\pi}{L}\mathbf{m}\right) = \frac{1}{(2\pi)^d} \sum_{\mathbf{m} \in \mathbb{Z}^d} \exp(i\mathbf{q} \cdot \mathbf{m}L) \tag{57}$$

Using this together with the results of the previous section, one can easily solve for $\tilde{f}(s)$, the Laplace transform of the squared inverse magnetization. The result is equivalent to the replacement

$$I_\Lambda\left(\frac{s}{2\lambda}\right) \to I_\Lambda\left(\frac{s}{2\lambda}\right) + \sum_{\mathbf{m} \in \mathbb{Z}^d \setminus \{0\}} \int_{\mathbf{q}}^{\Lambda} \frac{e^{i\mathbf{q} \cdot \mathbf{m}L}}{(q^2 + s/2\lambda)} \tag{58}$$

in (40). From it the asymptotic form of $\tilde{f}(s)$ follows in a straightforward fashion. Ignoring corrections to scaling and setting again the momentum unit μ (or Λ) equal to one, we find

$$\tilde{f}_{\mathrm{as}}(s) = (2\lambda A_\varepsilon)^{-1} L^{4-\varepsilon} \frac{D_u/L^2[m^{(i)}]^2 + 2\lambda/sL^2}{h(sL^2/2\lambda)} \tag{59}$$

where

$$h(x) = x^{1-\varepsilon/2} + [\Gamma(-1+\varepsilon/2)]^{-1} \int_0^\infty d\tau\, \tau^{-2+\varepsilon/2} g(\tau)\, e^{-x\tau} \tag{60}$$

with

$$g(\tau) = \left(\sum_{n=-\infty}^{\infty} e^{-n^2/4\tau} \right)^d - 1 \tag{61}$$

According to the phenomenological scaling theory described in Section 2, $\tilde{f}(s)$ should have a scaling form on sufficiently long length and time scales. As can be easily deduced from the scaling form (28) of the magnetization, we should have

$$\tilde{f}_{\mathrm{as}}(s) = \mathrm{const} \cdot [m^{(i)}]^{-(z+2\beta/\nu)/x_i} \mathscr{F}(st_i, t_i/t_L) \tag{62}$$

where $\mathscr{F}$ is related to the function $\mathscr{X}_c$ via

$$\mathscr{F}(\sigma, \rho) = \int_0^\infty d\vartheta\, e^{-\sigma\vartheta} [\mathscr{X}_c(\vartheta, \rho)]^{-2} \tag{63}$$

Recalling that the exponent $z + 2\beta/\nu$ takes the value $4-\varepsilon$ for $n=\infty$, we see that our result (59) has the anticipated form. We fix the nonuniversal scales as follows. First, we keep our previous choice (21c) of t_i. Further, the amplitude of $m_c(t)$ is normalized as in the bulk case; that is, we require that the scaling function $\mathscr{X}_c(\vartheta_i, \rho)$ one obtains through inversion of (63) reduces for $\rho = 0$ exactly to our previous result (56) for the bulk scaling function $\mathscr{X}_{bc}$. Finally, we wish to choose the time scale t_L such that $m_c(t) \sim \exp(-t/t_L)$ for $t \gg t_L \gg t_i$. As we shall see, this leads to

$$t_L = L^2/x_0\lambda \tag{64}$$

where x_0 is the positive real zero of the function $h(x)$. The so-defined finite-size linear relaxation time t_L is d dependent on account of the d dependence of x_0, whose numerical values for $2 < d < 4$ are depicted in Fig. 1. The d dependence is completely consistent with previous works[14,17] in which the relaxation time was calculated from the static correlation length. In particular, one obtains $t_L \sim \varepsilon^{-1/2}$ for $\varepsilon \to 0^+$ and $t_L \sim 1/(d-2)$ for $d \to 2^+$.

With this choice of nonuniversal factors the scaling function takes the form

$$\mathscr{F}(\sigma, \rho) = \Gamma\left(2 - \frac{\varepsilon}{2}\right)\left(\frac{2\rho}{x_0}\right)^{-1+\varepsilon/2} \frac{2/(2-\varepsilon) + 1/\sigma}{h(x_0\sigma/2\rho)} \tag{65}$$

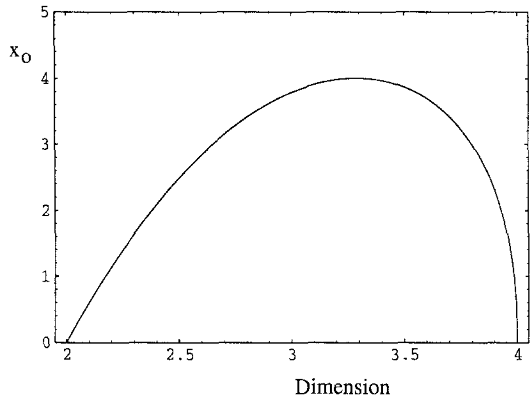

Fig. 1. Numerical result for x_0 as a function of d. The curve reflects the d dependence of the inverse of the linear relaxation time: $t_L = L^2/x_0\lambda$.

The backtransform of (65)—and hence the scaling function $\mathscr{X}_c$—cannot be represented in terms of standard functions. However, employing well-known results about Laplace transformations,[23] one can gain valuable analytic information about the asymptotic behavior of $\mathscr{X}_c(\vartheta_i, \rho)$, both for small ϑ_i as well as for large ϑ_i. This in turn can be compared with the results of the phenomenological analysis presented at the end of Section 3. Crucial for the determination of the asymptotic behavior is the analytic structure of $\mathscr{F}(\sigma, \rho)$ at fixed ρ as a function of complex σ. For our purposes here, it turns out sufficient to know that $\mathscr{F}$ is meromorphic in the right half-plane $\mathrm{Re}\,\sigma > 0$ and has a first-order pole at the real, positive root $\sigma_0 = 2\rho$ corresponding to the zero x_0 of $h(x = x_0\sigma/2\rho)$.

Consider first the limit $\vartheta_i \to 0^+$ at fixed ρ. The asymptotic form of the scaling function $\mathscr{X}_c(\vartheta_i, \rho)$ in this limit can be obtained from the $\sigma \to \infty$ limit of $\mathscr{F}$. The latter function is regular in the half-plane $\mathrm{Re}\,\sigma > \sigma_0$ and, as $\sigma \to \infty$ in this region, behaves as

$$\mathscr{F}(\sigma, \rho) \underset{\sigma \to \infty}{\approx} \Gamma(1 - \varepsilon/2)\, \sigma^{-1+\varepsilon/2}[1 + O(\rho\sigma, \sigma)] \tag{66}$$

From the Laplace backtransform of the right-hand side of this equation we obtain the limiting form

$$\mathscr{X}_c(\vartheta_i, \rho) \underset{\vartheta_i \to 0^+}{\approx} \vartheta_i^{\varepsilon/4}[1 + O(\rho\vartheta_i, \vartheta_i)] \tag{67}$$

which is in accordance with the bulk result (56). Note that the limiting forms (66) and (67) are independent of ρ. However, the leading corrections include terms $\propto \rho$, so that the width of the region in which the asymptotic

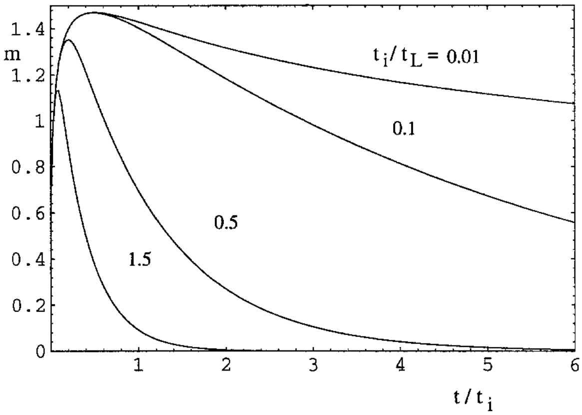

Fig. 2. The magnetization as a function of the scaled time, t/t_i, for $d=3$ and various values of t_i/t_L.

behavior is observed shrinks as ρ increases beyond 1, i.e., as t_L becomes smaller than t_i. In Fig. 2 we present results for $m_c(t)$ as a function of $\vartheta_i = t/t_i$ for various values of $\rho = t_i/t_L$. These results, which were obtained by numerical Laplace inversion, bear out this shrinking. In accordance with our analytical result (67), they also show that $m_c \sim t^{\varepsilon/4}$ as long as t is much smaller than t_i and t_L. This is again the initial-slip behavior, as expected from our phenomenological analysis.

Consider next the other asymptotic case, $\vartheta_i \to \infty$. In this limit the behavior of $\mathscr{Z}_c(\vartheta_i, \rho)$ is governed by the pole $\sigma_0 = 2\rho$ of $\mathscr{F}$, i.e., by the singularity farthest to the right of the Im σ axis. To see this, let a and b be two real numbers with $0 < a < \sigma_0 < b$, where a can be chosen arbitrarily close to zero. Within the strip $a \leqslant \operatorname{Re} \sigma \leqslant b$, the function $\mathscr{F}$ can be represented as a Laurent series

$$\mathscr{F}(\sigma, \rho) = \frac{\varphi(\rho)}{\sigma - 2\rho} + \text{regular terms} \tag{68}$$

Here

$$\varphi(\rho) = \Gamma\left(2 - \frac{\varepsilon}{2}\right)\left(\frac{2\rho}{x_0}\right)^{\varepsilon/2} \frac{2/(2-\varepsilon) + 1/2\rho}{h'(x_0)} \tag{69}$$

where the prime denotes the derivative. The regular terms can be calculated but will not be important for the asymptotic behavior considered. Further, when $\sigma = \sigma' + i\sigma''$ with $\sigma' \in [a, b]$, then $\mathscr{F}$ tends uniformly to zero as $\sigma'' \to \pm\infty$. In computing the Laplace inversion $\int_{\mathscr{C}} \mathscr{F}\, e^{\sigma \vartheta_i}\, d\sigma/2\pi i$, we may therefore[23] deform the standard contour $\mathscr{C} \equiv \mathscr{C}_b : \sigma = b + i\sigma''$, $-\infty < \sigma'' < +\infty$, into an analogous path $\mathscr{C}_a$ with $\sigma = a + i\sigma''$ plus a closed contour that encircles the pole σ_0 once and has arbitrarily small diameter. The contribution from the closed contour dominates in the limit $\vartheta_i \to \infty$, since the contribution from $\mathscr{C}_a$ is smaller by a factor $e^{-(\sigma_0 - a)\vartheta_i}$. It follows that the scaling function $\mathscr{X}_c$ behaves as

$$\mathscr{X}_c(\vartheta_i \rho) \underset{\vartheta_i \to \infty}{\approx} [\varphi(\rho)]^{-1/2} e^{-\rho \vartheta_i} \tag{70}$$

The implied asymptotic form of $m_c(t)$ for large t may be written as

$$m_c(t) \underset{t \to \infty}{\approx} \text{const} \cdot m^{(i)} t_L^{\varepsilon/4} [4/(2-\varepsilon) + t_L/t_i]^{-1/2} e^{-t/t_L} \tag{71}$$

where the exponent $\varepsilon/4$ of t_L may again be recognized as the $n = \infty$ value of θ'.

It should be emphasized that the limiting form (71) applies for all values of t_i/t_L as long as $t \gg t_L$. In other words, for such long times $m_c(t)$ decays exponentially, where the amplitude of the exponential depends on the ratio t_i/t_L while its time scale—namely, the finite-size linear relaxation time t_L—does not. In the limit $t_i/t_L \to 0$ (corresponding to case I of Section 3), all dependence contained in the amplitude drops out and (71) reduces to

$$m_c(t) \approx t_L^{-\beta/\nu z} e^{-t/t_L} \qquad (\text{for } t \gg t_L \gg t_i) \tag{72}$$

In the opposite limit $t_i/t_L \to \infty$ (corresponding to case II of Section 3), we find

$$m_c(t) \approx m^{(i)} t_L^{\theta'} e^{-t/t_L} \qquad (\text{for } t \gg t_i \gg t_L) \tag{73}$$

Hence the exponent θ' governing the bulk initial-slip behavior can also be deduced from the amplitude of the exponential in this asymptotic regime. For intermediate values of t_i/t_L the amplitude of the exponential displays a smooth crossover between the above two limiting forms.

The numerical backtransforms $m_c(t)$ presented in Fig. 2 corroborate the analytical results derived in this section. In particular, they show the crossover at fixed t_i/t_L from the initial-slip behavior at short times to the exponential decay at long times. Likewise they visualize the crossover from bulk to finite-size behavior with the shrinking of the initial-slip regime as t_i/t_L increases. Thus for $t_i/t_L = 0.01$ the magnetization in Fig. 2 decays as $t^{-1/4}$ in the regime $2 \lesssim t/t_i \leqslant 6$. As t_i/t_L increases, the decay becomes more and more exponential, and the crossover from initial-slip to linear relaxation behavior is shifted to smaller values of t/t_i.

6. SUMMARY AND CONCLUSIONS

We considered the critical dynamics of finite systems evolving from a nonequilibrium state with initial values $m(t=0)=m^{(i)}$ of the order parameter $m(t)$. In order to describe the critical relaxation of the order parameter toward thermal equilibrium for times t long compared to the microscopic scale t_{mic}, we developed a scaling theory into which the initial condition $m(0)=m^{(i)}$ is properly built. If the relaxation process takes place directly at the bulk critical point $T=T_c$, $H=0$, so that the bulk linear relaxation time $t_\tau \sim \tau^{-\nu z}$ is infinite, then the theory involves two characteristic time scales: the usual finite-size linear relaxation time $t_L \sim L^z$ and an initial-value time $t_i \sim [m^{(i)}]^{-z/x_i}$. The important point is that x_i, the scaling dimension of $m^{(i)}$, is an *independent scaling index* that, in general, *differs from the scaling dimension* $x_\phi = \beta/\nu$ *of the equilibrium bulk order parameter*, as was first shown by Janssen *et al.* in their pioneering work[7] on critical relaxation in bulk systems.

Previous theories of critical relaxation in finite systems either focused from the outset on the regime $t \gg t_i$ in which all dependence on the initial condition has faded away, or tacitly assumed that $x_i = x_\phi$. As a consequence, the time regime $t_{\mathrm{mic}} \ll t \lesssim t_i$ with universal dependence on the initial condition was missed or incorrectly treated.

We investigated the problem first by formulating a phenomenological scaling theory and then corroborated the general aspects of our findings through an explicit solution of the time-dependent n-component Ginzburg–Landau model with nonconserved order parameter in the many-component limit $n \to \infty$. This model was well suited for our purposes since the necessary calculations could be carried out mostly analytically but nevertheless yielded nontrivial results which one expects to subsist qualitatively also for the general n-vector model.

The fact that the exponent $\theta' = (x_i - x_\phi)/z$ is positive rather than zero led to nontrivial short-time behavior of the order parameter for $t_{\mathrm{mic}} \ll t \ll t_i$, as well as to interesting crossover effects. In the regime $t_{\mathrm{mic}} \ll t \ll t_i \ll t_L$ we recovered the expected initial-slip behavior $m(t) \sim t^{\theta'}$ known from Janssen *et al.*'s bulk analysis. For $t \gg t_L \gg t_{\mathrm{mic}}$ we found the usual exponential decay $m(t) \sim \exp(-t/t_L)$ of finite-size linear relaxation behavior. However, the amplitude of the exponential turned out to depend on the initial value $m^{(i)}$ and to vary in an interesting fashion as the ratio t_i/t_L is changed. As can be read off from the limiting forms (72) and (73), the amplitude crosses over from a dependence of the form $\sim m^{(i)} t_L^{\theta'}$ for $t_i \ll t_L$ to a behavior $\sim t_L^{-\beta/\nu z}$ for $t_i \gg t_L$. These results clearly indicate that the process of critical relaxation in finite systems is much richer that previously thought. Of course, the various types of asymptotic behaviors predicted above call for detailed verification by means of Monte Carlo simulations such as in ref. 24.

There are several obvious directions in which our work should be extended. First, the analysis should also be carried out for the general n-vector model. The way to do this is to integrate out the $\mathbf{q} \neq 0$ modes to some finite order of renormalization-group improved perturbation theory. This procedure leads to a nonlinear stochastic differential equation for the spatially constant mode which can be solved numerically.[15,17] Second, other boundary conditions should be considered. For example, one might take free boundary conditions on the faces bounding the system in one or several principal directions while keeping periodic boundary conditions along the remaining directions. As a natural further generalization one would like to allow surface magnetic fields and modified (either enhanced or weakened) bonds on the bounding surfaces. Moreover, one would like to consider other geometries and topologies such as film geometries or finite systems with curved boundaries. Last but not least, more complicated models belonging to other universality classes of critical dynamics should be investigated.

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